

NONLINEAR SCHRÖDINGER EQUATION WITH A WHITE-NOISE POTENTIAL: PHASE-SPACE APPROACH TO SPREAD AND SINGULARITY

ALBERT C. FANNJIANG

ABSTRACT. We propose a phase-space formulation for the nonlinear Schrödinger equation with a white-noise potential in order to shed light on two issues: the rate of spread and the singularity formation in the average sense. Our main tools are the energy law and the variance identity. The method is completely elementary.

For the problem of wave spread, we show that the ensemble-averaged dispersion in the critical or defocusing case follows the cubic-in-time law while in the supercritical and subcritical focusing cases the cubic law becomes an upper and lower bounds respectively.

We have also found that in the critical and supercritical focusing cases the presence of a white-noise random potential results in different conditions for singularity-with-positive-probability from the homogeneous case but does not prevent singularity formation. We show that in the supercritical focusing case the ensemble-averaged self-interaction energy and the momentum variance can exceed any fixed level in a finite time with positive probability.

1. INTRODUCTION

High-intensity laser beams propagating in dielectric media such as optical fibers, films or air are important problems both from fundamental and practical perspectives. The physical process is nonlinear and the amplitude modulation Ψ is customarily described by a nonlinear Schrödinger equation with an additional inhomogeneous potential V representing the random impurities in the medium.

Let z and $\mathbf{x} \in \mathbb{R}^d$ be the longitudinal and transverse coordinates of the wave beam. In the physical setting, the transverse dimension d may be 0, 1, 2. The simplest non-dimensional form of the nonlinear Schrödinger (NLS) equation with a random potential V then reads [11]

$$(1) \quad i \frac{\partial}{\partial z} \Psi + \frac{\gamma}{2} \Delta_{\mathbf{x}} \Psi + \gamma^{-1} g |\Psi|^{2\sigma} \Psi + \gamma^{-1} V \circ \Psi = 0, \quad \sigma > 0.$$

Similar equations also arise in many other contexts such as in plasma physics and the Bose-Einstein condensation, see [13] and references therein. Here we have written the equation in the comoving reference frame at the group velocity and non-dimensionalized the equation with the longitudinal and transverse reference lengths L_z, L_x ; $\gamma = L_z/(kL_x^2)$ is the Fresnel number; g is nonlinear coupling coefficient with $g > 0$ representing the self-focusing case and $g < 0$ the self-defocusing case. For a nonlinear Kerr medium $\sigma = 1$ leading to the cubic NLS equation. We shall let σ be an arbitrary positive constant. We shall consider white-noise-in- z , \mathbf{x} -homogeneous random potential V with

$$\langle V(z, \mathbf{x}) V(z', \mathbf{x}') \rangle = \delta(z - z') \int e^{i(\mathbf{x} - \mathbf{x}') \cdot \mathbf{p}} \Phi(\mathbf{p}) d\mathbf{p}$$

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where $\Phi(\mathbf{p})$ is the power spectral density and the product $V \circ \Psi$ in eq. (1) stands for the Stratonovich product. Here and below the bracket $\langle \cdot \rangle$ denotes the ensemble average w.r.t. the random medium. In view of the real-valuedness of V we may assume

$$(2) \quad \Phi(-\mathbf{p}) = \Phi(\mathbf{p}), \quad \forall \mathbf{p} \in \mathbb{R}^d$$

Such a short-range-in- z potential arises in long-distance propagation when the longitudinal length L_z of the wave beam is much larger than the correlation length of the random impurities in the medium resulting in rapidly fluctuating potential in the non-dimensionalized coordinates. And it is well known that in the short-range-correlation scaling limit a non-white-in- z multiplicative noise gives rise to a Stratonovich, instead of Itô, integral with respect to a white-in- z noise [12].

We are particularly interested in the problems of wave spread and singularity formation. To this end we shall use the phase-space approach of the Wigner distribution which is particularly useful in the regime of low Fresnel number $\gamma \ll 1$ which can be viewed as a high-frequency limit. The main ingredient of our analysis is the phase-space variance identity by which we derive various estimates for the mean spread, defined in (9), including precise behavior in the defocusing $g \leq 0$ or critical case $d\sigma = 2$. To our knowledge, these are significant improvement over previous results (e.g. [1], [14], [3]) which are mostly for the linear problem. Wave spread is an important physical quantity in itself and in technological applications such as estimating the effect of nonlinearity on the information capacity of optical fiber communications [8], [10].

Using the variance identity we further show that the scattering by the random potentials results in conditions for singularity-with-positive-probability that are different from those for the homogeneous case.

In the present setting the physical roles of t and z are interchanged: the variable t is space-like while the variable z is time-like and we will refer to it as “time” occasionally, especially in the discussion of finite-time singularity.

2. WIGNER DISTRIBUTION

We consider the Wigner distribution of the form

$$(3) \quad W(z, \mathbf{x}, \mathbf{p}) = \frac{1}{(2\pi)^d} \int e^{-i\mathbf{p} \cdot \mathbf{y}} \Psi \left(z, \mathbf{x} + \frac{\gamma \mathbf{y}}{2} \right) \Psi^* \left(z, \mathbf{x} - \frac{\gamma \mathbf{y}}{2} \right) d\mathbf{y}$$

which is always real-valued and may be thought of as quasi-probability density on the phase space [9], [6]. First its marginals are essentially the position and momentum densities

$$\begin{aligned} \int W(z, \mathbf{x}, \mathbf{p}) d\mathbf{p} &= |\Psi|^2(z, \mathbf{x}) \equiv \rho(z, \mathbf{x}) \\ \frac{1}{(2\pi)^d} \int W(z, \mathbf{x}, \mathbf{p}) d\mathbf{x} &= \frac{1}{\gamma^d} |\hat{\Psi}(z, \mathbf{p})|^2 \end{aligned}$$

which are nonnegative; indeed any marginal on a d -dimensional subspace is nonnegative everywhere. Also, the energy flux is given by

$$(4) \quad \frac{1}{2i} (\Psi \nabla \Psi^* - \Psi^* \nabla \Psi) = \int_{\mathbb{R}^d} \mathbf{p} W(z, \mathbf{x}, \mathbf{p}) d\mathbf{p}.$$

In the whole phase space, however, the Wigner distribution is not everywhere non-negative in general and hence can not be a true probability density. As $\gamma \rightarrow 0$, the Wigner distribution has a nonnegative-measure-valued weak limit, called the Wigner measure [9]. Therefore the Wigner distribution is particularly useful for analyzing high frequency behaviors of waves.

3. THE EVOLUTION EQUATIONS

From the NLS eq. (1) it is straightforward to derive the closed-form (Wigner-Moyal) equation for the Wigner distribution in the Itô sense [4], [5]

$$(5) \quad dzW + \mathbf{p} \cdot \nabla_{\mathbf{x}} W dz + \mathcal{U}_\gamma W dz - \mathcal{Q}_\gamma W dz + \mathcal{V}_\gamma W dz = 0.$$

Here the *self-adjoint* operator \mathcal{Q}_γ is the Stratonovich correction term

$$\begin{aligned} \mathcal{Q}_\gamma W(z, \mathbf{x}, \mathbf{p}) &= \int \Phi(\mathbf{q}) \gamma^{-2} [-2W(z, \mathbf{x}, \mathbf{p}) + W(z, \mathbf{x}, \mathbf{p} - \gamma \mathbf{q}) + W(z, \mathbf{x}, \mathbf{p} + \gamma \mathbf{q})] d\mathbf{q} \end{aligned}$$

and \mathcal{U}_γ and \mathcal{V}_γ are the nonlinear and linear Moyal operators, respectively

$$\begin{aligned} \mathcal{U}_\gamma W(z, \mathbf{x}, \mathbf{p}) &= i \int e^{i\mathbf{q} \cdot \mathbf{x}} \gamma^{-1} [W(z, \mathbf{x}, \mathbf{p} + \gamma \mathbf{q}/2) - W(z, \mathbf{x}, \mathbf{p} - \gamma \mathbf{q}/2)] \hat{U}(z, \mathbf{q}) d\mathbf{q}, \\ \mathcal{V}_\gamma W(z, \mathbf{x}, \mathbf{p}) &= i \int e^{i\mathbf{q} \cdot \mathbf{x}} \gamma^{-1} [W(z, \mathbf{x}, \mathbf{p} + \gamma \mathbf{q}/2) - W(z, \mathbf{x}, \mathbf{p} - \gamma \mathbf{q}/2)] \hat{V}(dz, d\mathbf{q}) \end{aligned}$$

with

$$U = g\rho^\sigma.$$

Here we have somewhat abused the notation by writing \hat{U} as the Fourier-transform-in- \mathbf{x} of the function $U(z, \cdot)$ and $\hat{V}(dz, d\mathbf{q})$ as the spectral measure of the white-noise potential $V(z, \cdot)$. The spectral measure $\hat{V}(dz, d\mathbf{p})$ is related to the power spectral density Φ of V as

$$\langle \hat{V}(dz, d\mathbf{p}) \hat{V}^*(dz', d\mathbf{q}) \rangle = \delta(z - z') \delta(\mathbf{p} - \mathbf{q}) \Phi(\mathbf{p}) dz d\mathbf{p} d\mathbf{q}.$$

An important property of these integral operators is

$$(6) \quad \int \mathcal{Q}_\gamma W d\mathbf{p} = \int \mathcal{U}_\gamma W d\mathbf{p} = \int \mathcal{V}_\gamma W d\mathbf{p} = 0$$

which will be useful in deriving the energy law and the variance identity.

A major advantage of formulating the wave process on the phase space using the Wigner distribution is that the (high-frequency) low-Fresnel number limit γ can be easily obtained. Formally we see that as $\gamma \rightarrow 0$

$$\begin{aligned} \mathcal{U}_\gamma W(z, \mathbf{x}, \mathbf{p}) &\rightarrow \mathcal{U}_0 W(z, \mathbf{x}, \mathbf{p}) \equiv \nabla_{\mathbf{x}} U(z, \mathbf{x}) \cdot \nabla_{\mathbf{p}} W(z, \mathbf{x}, \mathbf{p}) \\ \mathcal{V}_\gamma W(z, \mathbf{x}, \mathbf{p}) &\rightarrow \mathcal{V}_0 W(z, \mathbf{x}, \mathbf{p}) \equiv \nabla_{\mathbf{x}} V(z, \mathbf{x}) \cdot \nabla_{\mathbf{p}} W(z, \mathbf{x}, \mathbf{p}) \\ \mathcal{Q}_\gamma W(z, \mathbf{x}, \mathbf{p}) &\rightarrow \mathcal{Q}_0 W(z, \mathbf{x}, \mathbf{p}) = \nabla_{\mathbf{p}} \cdot \mathbf{D} \nabla_{\mathbf{p}} W(z, \mathbf{x}, \mathbf{p}) \end{aligned}$$

with the diffusion coefficient

$$(7) \quad \mathbf{D} = \int \Phi(\mathbf{p}) \mathbf{p} \otimes \mathbf{p} d\mathbf{p}.$$

We shall refer to eq. (5) as the nonlinear Wigner-Moyal -Itô (NWMI) equation when $\gamma > 0$ and as the nonlinear Liouville-Itô (NLI) equation when $\gamma = 0$.

Another advantage of working with eq. (5) is that one can use it to evolve the mixed-state initial condition. The mixed-state Wigner distribution is a convex combination of the pure-state Wigner distributions (3) described as follows.

Let $\{\Psi_\alpha\}$ be a family of L^2 functions parametrized by α which is weighted by a probability measure $P(d\alpha)$. Denote the pure-state Wigner distribution (3) by $W[\Psi_\alpha]$. A mixed-state Wigner distribution is given by

$$(8) \quad \int W[\Psi_\alpha] P(d\alpha).$$

The limiting set, as $\gamma \rightarrow 0$, of the mixed state Wigner distributions constitute the nonnegative Wigner measures [6], [9]. The NWMI or NLI equation preserves the mixed-state form (8).

3.1. Basic properties. The NWMI and NLI equations conserve the total mass, i.e.

$$\frac{\partial}{\partial z} N = 0, \quad N = \int W(z, \mathbf{x}, \mathbf{p}) d\mathbf{x} d\mathbf{p}$$

and the L^2 norm

$$\frac{\partial}{\partial z} \int W^2(z, \mathbf{x}, \mathbf{p}) d\mathbf{x} d\mathbf{p} = 0.$$

One can absorb the effect of the total mass N into g by the obvious rescaling of W in the eq. (5). Henceforth we assume that $N = 1$.

Let S_x and S_p be the spreads (or variances) of position and momentum, respectively

$$(9) \quad S_x(z) = \int |\mathbf{x}|^2 W(z, \mathbf{x}, \mathbf{p}) d\mathbf{x} d\mathbf{p}$$

$$(10) \quad S_p(z) = \int |\mathbf{p}|^2 W(z, \mathbf{x}, \mathbf{p}) d\mathbf{x} d\mathbf{p}.$$

A natural space of initial data and solutions is the subspace $\mathcal{W} \subset L^2(\mathbb{R}^{2d})$ with a finite Dirichlet form

$$- \int W \mathcal{Q}_\gamma W d\mathbf{x} d\mathbf{p} < \infty$$

and finite, positive (pre-ensemble-averaged) variances $S_x \in (0, \infty), S_p \in (0, \infty)$. In addition, we shall also assume that the initial data have a finite Hamiltonian $H \in (-\infty, \infty)$ with

$$(11) \quad H = \frac{1}{2} S_p - \frac{g}{\sigma + 1} \int \rho^{\sigma+1} d\mathbf{x}, \quad \rho(z, \mathbf{x}) = \int W(z, \mathbf{x}, \mathbf{p}) d\mathbf{p}.$$

The first term S_p in H is the kinetic energy and the second term is the (self-interaction) potential energy. Note that in the presence of a random potential, the value of the Hamiltonian (11) is not conserved under the evolution (5). Indeed, the ensemble-averaged value of the Hamiltonian increases in time, cf. (29) below.

4. ENERGY AND VARIANCE IDENTITY

Let us analyze the evolution of the mean Hamiltonian $\langle H \rangle$ and the average spread $\langle S_x \rangle$. First note the result of \mathcal{U}_γ acting on the quadratic polynomials:

$$(12) \quad \mathcal{U}_\gamma \mathbf{x} = 0$$

$$(13) \quad \mathcal{U}_\gamma \mathbf{p} = i \int e^{i\mathbf{q} \cdot \mathbf{x}} \mathbf{q} \hat{U}(\mathbf{q}) d\mathbf{q} = \nabla_{\mathbf{x}} U$$

$$(14) \quad \mathcal{U}_\gamma |\mathbf{x}|^2 = 0$$

$$(15) \quad \mathcal{U}_\gamma \mathbf{x} \cdot \mathbf{p} = i \int e^{i\mathbf{q} \cdot \mathbf{x}} \mathbf{x} \cdot \mathbf{q} \hat{U}(\mathbf{q}) d\mathbf{q} = \mathbf{x} \cdot \nabla_{\mathbf{x}} U$$

$$(16) \quad \mathcal{U}_\gamma |\mathbf{p}|^2 = i \int e^{i\mathbf{q} \cdot \mathbf{x}} 2\mathbf{p} \cdot \mathbf{q} \hat{U}(\mathbf{q}) d\mathbf{q} = 2\mathbf{p} \cdot \nabla_{\mathbf{x}} U.$$

A remarkable fact is that these results are independent of $\gamma \geq 0$.

Using (12)-(16) and (6) we obtain by integrating eq. (5) that

$$\begin{aligned} \frac{\partial}{\partial z} \langle S_p \rangle &= \left\langle \int \nabla_{\mathbf{x}} U \cdot 2\mathbf{p} W d\mathbf{x} d\mathbf{p} \right\rangle + \int \mathcal{Q}_\gamma |\mathbf{p}|^2 \langle W \rangle d\mathbf{p} d\mathbf{x} \\ \frac{\partial}{\partial z} \frac{g}{\sigma + 1} \int \langle \rho^{\sigma+1} \rangle d\mathbf{x} &= \left\langle \int \nabla_{\mathbf{x}} U \cdot \mathbf{p} W d\mathbf{p} d\mathbf{x} \right\rangle. \end{aligned}$$

from which the evolution equation for the mean value of the Hamiltonian follows

$$(17) \quad \frac{\partial}{\partial z} \langle H \rangle = \int \mathcal{Q}_\gamma |\mathbf{p}|^2 \langle W \rangle d\mathbf{p} d\mathbf{x}.$$

We shall refer to (17) as the *energy law*.

4.1. Variance identity. The variance identity has been long used to derive the wave collapse condition for the NLS in the homogeneous case [13]. Below we reformulate it for the phase space evolution eq. (5).

We have another set of remarkably simple properties of \mathcal{Q}_γ independent of $\gamma \geq 0$:

$$(18) \quad \mathcal{Q}_\gamma \mathbf{x} = 0$$

$$(19) \quad \mathcal{Q}_\gamma \mathbf{p} = \int \Phi(\mathbf{p} - \mathbf{q})(\mathbf{p} - \mathbf{q}) d\mathbf{q} = 0 \quad (\text{by (2)})$$

$$(20) \quad \mathcal{Q}_\gamma |\mathbf{x}|^2 = 0$$

$$(21) \quad \mathcal{Q}_\gamma \mathbf{x} \cdot \mathbf{p} = \mathbf{x} \cdot \int \Phi(\mathbf{p} - \mathbf{q})(\mathbf{p} - \mathbf{q}) d\mathbf{q} = 0 \quad (\text{by (2)})$$

$$(22) \quad \mathcal{Q}_\gamma |\mathbf{p}|^2 = - \int \Phi(\mathbf{p} - \mathbf{q}) [|\mathbf{p}|^2 - |\mathbf{q}|^2] d\mathbf{q}$$

$$(23) \quad = - \int \Phi(\mathbf{q})(2\mathbf{p} - \mathbf{q}) \cdot \mathbf{q} d\mathbf{q}$$

$$(24) \quad = \int \Phi(\mathbf{q}) |\mathbf{q}|^2 d\mathbf{q} \equiv 2R.$$

For the diffusion operator \mathcal{Q}_0 we have $R = 2 \times \text{tr}(\mathbf{D})$ where the diffusion matrix \mathbf{D} is given by (7). We shall use the above identities to perform integrating by parts in the derivation of the variance identity.

Combining the results from the previous section and (18)-(24) we obtain the rate of change of $\langle S_x \rangle$

$$(25) \quad \frac{\partial}{\partial z} \langle S_x \rangle = 2 \langle S_{xp} \rangle$$

where S_{xp} is the cross-moment

$$S_{xp} = \int \mathbf{x} \cdot \mathbf{p} W d\mathbf{x} d\mathbf{p}.$$

Differentiating S_{xp} and taking expectation we obtain

$$\frac{\partial}{\partial z} \langle S_{xp} \rangle = \langle S_p \rangle - \frac{gd\sigma}{\sigma+1} \int \langle \rho^{\sigma+1} \rangle d\mathbf{x} d\mathbf{p}$$

Hence the second derivative of S_x becomes

$$(26) \quad \frac{\partial^2}{\partial z^2} \langle S_x \rangle = 4 \langle H \rangle + \frac{2(2-d\sigma)g}{\sigma+1} \int \langle \rho^{\sigma+1} \rangle d\mathbf{x}.$$

Alternatively using the definition of H we can rewrite the variance identity (26) as

$$(27) \quad \frac{\partial^2}{\partial z^2} \langle S_x \rangle = 2d\sigma \langle H \rangle + (2-d\sigma) \langle S_p \rangle$$

which in the critical case $d\sigma = 2$ becomes

$$(28) \quad \frac{\partial^2}{\partial z^2} \langle S_x \rangle = 2d\sigma \langle H \rangle.$$

Both (26) and (27) will be useful for estimating wave spread.

5. DISPERSION RATE

Although the medium is lossless, reflected in the fact that the total mass $N = 1$ is conserved, the value of the Hamiltonian, however, is not conserved by the evolution since the random scattering is not elastic due to time-varying nature of the random potential. Indeed, by (17) and (24), the average Hamiltonian is an increasing function of time

$$(29) \quad \frac{\partial}{\partial z} \langle H \rangle = R, \quad \langle H \rangle(z) = H(0) + Rz$$

due to the diffusion-like spread in the momentum \mathbf{p} .

In the critical case $d\sigma = 2$, we obtain from eq. (28) and (29) the exact result

$$(30) \quad \frac{\partial^2}{\partial z^2} \langle S_x \rangle = 4 \langle H \rangle = 4H(0) + 4Rz$$

before any singularity formation causes a possible breakdown of the variance identity. Integrating (30) twice we obtain the exact spread rate as stated below.

Proposition 1. *If $d\sigma = 2$ or $g = 0$, then*

$$(31) \quad \langle S_x \rangle(z) = S_x(0) + 2S_{xp}(0)z + 2H(0)z^2 + \frac{2R}{3}z^3.$$

The analogous result ($S_x \sim z^3$) for the linear Schrödinger equation ($d = 1, g = 0$) with a random potential has been proved previously [1], [3].

Next we consider the supercritical case and the defocusing case. We have from (26) that

$$(32) \quad \frac{\partial^2}{\partial z^2} \langle S_x \rangle = 4 \langle H \rangle + \frac{(4 - 2d\sigma)g}{\sigma + 1} \int \langle \rho^{\sigma+1} \rangle d\mathbf{x}$$

and hence

$$\begin{aligned} \frac{\partial^2}{\partial z^2} \langle S_x \rangle &\leq 4 \langle H \rangle = 4H(0) + 4Rz, \quad \text{for } g(2 - d\sigma) < 0 \\ \frac{\partial^2}{\partial z^2} \langle S_x \rangle &\geq 4 \langle H \rangle = 4H(0) + 4Rz, \quad \text{for } g(2 - d\sigma) \geq 0. \end{aligned}$$

On the other hand, from (27) we obtain for any g

$$\begin{aligned} \frac{\partial^2}{\partial z^2} \langle S_x \rangle &\leq 2d\sigma \langle H \rangle, \quad \text{for } 2 - d\sigma \leq 0 \\ \frac{\partial^2}{\partial z^2} \langle S_x \rangle &\geq 2d\sigma \langle H \rangle, \quad \text{for } 2 - d\sigma \geq 0. \end{aligned}$$

Integrating the above inequalities twice, we obtain the following.

Proposition 2. *The following estimates hold*

$$(33) \quad \langle S_x(z) \rangle \leq S_x(0) + 2S_{xp}(0)z + 2H(0)z^2 + \frac{2}{3}Rz^3, \quad g(2 - d\sigma) \leq 0$$

$$(34) \quad \langle S_x(z) \rangle \geq S_x(0) + 2S_{xp}(0)z + 2H(0)z^2 + \frac{2}{3}Rz^3, \quad g(2 - d\sigma) \geq 0$$

and

$$(35) \quad \langle S_x(z) \rangle \leq S_x(0) + 2S_{xp}(0)z + d\sigma H(0)z^2 + \frac{d\sigma}{3}Rz^3, \quad 2 \leq d\sigma$$

$$(36) \quad \langle S_x(z) \rangle \geq S_x(0) + 2S_{xp}(0)z + d\sigma H(0)z^2 + \frac{d\sigma}{3}Rz^3, \quad 2 \geq d\sigma.$$

Therefore

Corollary 1. *Assume $g < 0$ (hence $H \geq 0$). Then*

$$\langle S_x(z) \rangle \leq S_x(0) + 2S_{xp}(0)z + (d\sigma \vee 2)H(0)z^2 + \frac{d\sigma \vee 2}{3}Rz^3$$

and

$$(37) \quad \langle S_x(z) \rangle \geq S_x(0) + 2S_{xp}(0)z + (d\sigma \wedge 2)H(0)z^2 + \frac{d\sigma \wedge 2}{3}Rz^3.$$

In other words the variance $\langle S_x \rangle$ is cubic-in-time in the defocusing case. In the subcritical case the cubic law is a lower bound while in the supercritical case the cubic law is an upper bound.

6. FINITE-TIME SINGULARITY

Finite-time singularity for the critical or supercritical ($d\sigma \geq 2$) self-focusing NLS equation in the homogeneous case is a well known effect [13]. In this case the singularity is the blow-up type $S_p, \int |\rho|^{\sigma+1} \rightarrow \infty$. Here we call breakdown of the variance identity (26)-(27) or the energy law (17) as finite-time singularity and seek the sufficient conditions for singularity with positive probability. We show that in the supercritical case with additional assumptions the finite-time singularity is of the blow-up type in the sense that $\langle S_p \rangle, \int \langle \rho^{\sigma+1} \rangle$ tend to infinity. As such the blow-up phenomenon discussed here is not necessarily a sure event but rather an event of a positive probability that S_p and $\int \rho^{\sigma+1}$ can exceed any fixed level in a finite time.

For $g \geq 0, d\sigma \geq 2$ one can bound S_x as in the inequality (35)

$$(38) \quad \langle S_x(z) \rangle \leq S_x(0) + 2S_{xp}(0)z + d\sigma H(0)z^2 + \frac{d\sigma R}{3}z^3 \equiv F(z)$$

and as motivated by the homogeneous case we look for the conditions when $F(z)$ vanishes at a finite $z \geq 0$.

A sufficient condition for $F(z)$ to vanish at a finite positive z can be derived from that $F(z)$ takes a non-positive value $F(z_0) \leq 0$ at its local minimum point $z_0 > 0$. The local minimum point z_0 is given by

$$(39) \quad z_0 = \frac{-H(0) + \sqrt{H(0)^2 - 2RS_{xp}(0)/(d\sigma)}}{R}.$$

Therefore we are led to the following conditions for singularity.

Proposition 3. *Assume $d\sigma \geq 2, g > 0$. The solutions of the NWMI or NLI equation develop singularities with positive probability at a finite time $z_* \leq z_0$ given by (39) under the condition $F(z_0) \leq 0$ and either one of the following conditions*

$$(40) \quad S_{xp}(0) < 0$$

$$(41) \quad S_{xp}(0) > 0, \quad H(0) < -\sqrt{\frac{2RS_{xp}(0)}{d\sigma}}.$$

Remark 1. *Clearly, the condition $F(z_0) \leq 0$ requires $H(0)$ to be sufficiently below \bar{H} by allowing the self-interaction energy*

$$-\frac{g}{\sigma+1} \int \rho^{\sigma+1}(0) d\mathbf{x} d\mathbf{p}$$

to be sufficiently negative.

Remark 2. *Using, instead, inequality (33) one can obtain an alternative expression*

$$(42) \quad z_0 = \frac{-H(0) + \sqrt{H(0)^2 - RS_{xp}(0)}}{R}$$

and the corresponding conditions for singularity formation, namely either $S_{xp}(0) < 0$ or $S_{xp}(0) > 0, H(0) < -\sqrt{RS_{xp}(0)}$.

It may be the case that for a given initial condition the solutions develop singularity at different times depending on the realization of random potential. To analyze such an effect we need the probabilistic versions of variance identity and energy law which are much more involved.

6.1. Blow-up. We will follow the argument of [7] to show more explicitly the blow-up mechanism in the case with the supercritical, self-focusing nonlinearity and give a sharper bound on z_* under certain circumstances.

Proposition 4. *Suppose $d\sigma > 2, g > 0$. Then under the conditions (40) and*

$$(43) \quad \langle H(z_*) \rangle = H(0) + Rz_* \leq 0, \quad \text{with } z_* = \frac{2S_x(0)}{S_{xp}(0)(2-d\sigma)},$$

$\partial \langle S_x \rangle / \partial z$ and $\langle S_p \rangle$ blow up before or at a finite time z_ .*

Proof. Since blow-up is a local phenomenon, S_x is a poor indicator of its occurrence. A more relevant object to consider is S_p . From (27) it follows that before z_*

$$(44) \quad \frac{\partial^2}{\partial z^2} \langle S_x \rangle = 2 \frac{\partial}{\partial z} \langle S_{xp} \rangle \leq (2-d\sigma) \langle S_p \rangle < 0.$$

Since $\partial \langle S_x \rangle / \partial z = 2 \langle S_{xp} \rangle < 0$ by (44) and (40), $\partial \langle S_x \rangle / \partial z$ is a negative, decreasing function up to the time z_* . Hence we have

$$(45) \quad 0 \leq \langle S_{xp} \rangle^2 \leq \langle S_x \rangle \langle S_p \rangle \leq S_x(0) \langle S_p \rangle, \quad z \in [0, z_*]$$

which implies

$$(46) \quad \langle S_p \rangle \geq \frac{\langle S_{xp} \rangle^2}{S_x(0)}.$$

Let $A(z) = -\langle \partial S_x(z) / \partial z \rangle \geq 0, z \leq z_*$. We have from (44), (46) and (25) the differential inequality

$$(47) \quad \frac{\partial}{\partial z} A \geq CA^2, \quad C = \frac{d\sigma - 2}{4S_x(0)} > 0$$

which yields the estimate

$$A(z) \geq \frac{A(0)}{1 - zCA(0)}, \quad z < \frac{1}{CA(0)} = z_*$$

and thus the blow-up of $A(z)$ before or at z_* . This along with (46) then implies the blow-up of S_p at a finite time. \square

The preceding argument demonstrates clearly the blow-up mechanism, namely the quadratic growth property (47).

Remark 3. *It should be noted that, since $H(z_0) > 0$, the condition (43) implies that $z_* < z_0$, as given by either (39) or (42). Therefore, under the condition (43), z_* provides a sharper upper bound on the time of singularity than z_0 .*

Remark 4. *Since $\langle H(z) \rangle$ is bounded over compact sets of z , Proposition 4 implies the blow-up of the self-interaction energy $\int \langle \rho^{\sigma+1} \rangle$ in a finite time.*

7. CONCLUSION

We have presented an elementary approach for analyzing the nonlinear Schrödinger equation with a white-noise potential. We have focused on the ensemble-averaged quantities such as the variance identity and energy law in order to shed light on two problems: the rate of spread and the singularity formation.

We have shown that the ensemble-averaged spread in the critical or defocusing case follows the cubic-in-time law while in the supercritical and subcritical focusing cases the cubic law becomes an upper and lower bounds respectively. In a separate publication we will use these estimates to analyze the limitation on channel capacity in optical fibers due to self-phase and cross-phase modulations [8].

We have also found singularity conditions in the critical and supercritical focusing cases. And we show the finite-time singularity in the supercritical case is of the blow-up type. The singularity/blow-up discussed in the present paper is not necessarily a sure event but that of a positive probability.

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E-mail address: `cafannjiang@ucdavis.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, DAVIS 95616-8633